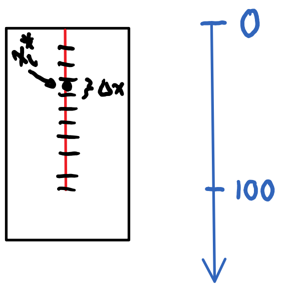
Lecture 29

Wednesday, November 16, 2016 8:04 AM

<u>Ex</u> A 200-lb cable is 100 ft long and hangs vertically from the top of a tall building. How much work is required to lift the cable to the top of the building?



Place origin at "the top of the building" and x-axis pointing downward.

- Divide cable into small parts of length Δx and pick a sample point x_i^* in the i^{th} subinterval.

- 100 ft and weighs 200 lbs.

  Weighs 2 lbs / ft

Then the <u>weight</u> of each part is $\left(\frac{2\text{lb}}{\text{ft}}\right) \cdot \Delta x \text{ ft}$

$= 2\Delta x$ lb = force on each piece.

Work done on the i^{th} part can be approximated by $(2\Delta x) \cdot x_i^*$

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} 2\Delta x \cdot x_i^* = \int_0^{100} 2x\,dx$$

$$= x^2 \Big]_0^{100} = 100^2 \text{ lb-ft} = 10000 \text{ lb-ft}$$

Average value of function

Let $y = f(x)$, $a \le x \le b$

Then the average value of f on $[a,b]$ is

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Ex Find the average value of $f(x) = \frac{\ln x}{x}$ on $[1,5]$.

$$f_{ave} = \frac{1}{5-1}\int_1^5 \frac{\ln x}{x}\,dx$$

Let $u = \ln x$

$$\frac{du}{dx} = \frac{1}{x} \Rightarrow du = \frac{1}{x}\,dx$$

$x = 1$, $u = \ln 1 = 0$

$x = 5$, $u = \ln 5$

$$f_{ave} = \frac{1}{4}\int_0^{\ln 5} u\,du = \frac{1}{4}\left[\frac{u^2}{2}\right]_0^{\ln 5}$$

$$= \frac{1}{4}\left[\frac{(\ln 5)^2}{2} - \frac{0^2}{2}\right] = \frac{1}{8}(\ln 5)^2$$

THE MEAN VALUE THEOREM FOR INTEGRALS

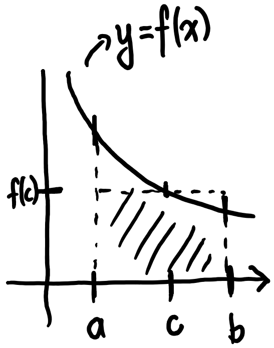
If f is continuous on $[a,b]$, then there exist a number c in $[a,b]$ s.t

$$f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\Rightarrow \boxed{f(c)(b-a) = \int_a^b f(x)\,dx}$$

GEOMETRIC INTERPRETATION



$\int_a^b f(x)\,dx \equiv$ Area under the curve

$f(c)(b-a) \equiv$ Area of rectangle

The MVT for integral states that for a positive function f, there exist a number c s.t the rectangle w/ base $[a,b]$ and height $f(c)$ has the same area as the region under f betn a & b ☒

<u>Ex</u> Let $f(x) = \frac{1}{x}$ on $[1,3]$.

Find c that satisfies the statement of MVT.

i.e $f(c) = f_{ave}$

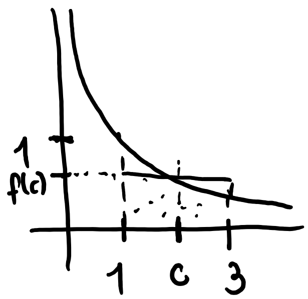
<u>Soln</u> $f_{ave} = \boxed{\frac{1}{3-1}}\int_1^3 \frac{1}{x}\,dx = \frac{1}{2}\Big[\ln|x|\Big]_1^3$

$f \quad = \frac{1}{2}[\ln 3 - \ln 1] = \frac{\ln 3}{}$

$$f_{ave} = \frac{1}{2}\left[\ln 3 - \ln 1\right] = \frac{\ln 3}{2}$$

Want to find c s.t $f(c) = f_{ave}$

$$\Rightarrow \frac{1}{c} = \frac{\ln 3}{2} \Rightarrow \boxed{c = \frac{2}{\ln 3}}$$



7.1 Integration by Parts

Product rule

$$\frac{d}{dx}\left[f(x) \cdot g(x)\right] = f(x) \cdot g'(x) + f'(x) \cdot g(x)$$

$$\Downarrow$$

$$\int f(x) \cdot g'(x) + f'(x)\, g(x)\, dx = f(x) \cdot g(x) + C$$

$\Downarrow$ Drop the C

$$\int f(x) \cdot g'(x)\, dx + \int f'(x) \cdot g(x)\, dx = f(x) \cdot g(x)$$

$$\Downarrow$$

$$\boxed{\int f(x) \cdot g'(x)\, dx = f(x) \cdot g(x) - \int f'(x)\, g(x)\, dx}$$

Formula for integration by parts

$u = f(x)$, $v = g(x)$

$\frac{du}{dx} = f'(x)$, $\frac{dv}{dx} = g'(x)$

$\Rightarrow du = f'(x)\,dx$, $dv = \underline{g'(x)}\,dx$

$$\boxed{\int u\,dv = u.v - \int v\,\underline{\underline{du}}} \rightarrow$$

$\underline{\text{Ex}}$ $\int x\cos x\,dx$

$\underline{u = x}$, $dv = \cos x\,dx \Rightarrow \frac{dv}{dx} = \cos x$

$\frac{du}{dx} = 1$ $\qquad \Rightarrow v = \int \cos x\,dx$

$\Rightarrow \underline{du = dx}$ $\qquad = \sin x$

$$\int u\,dv = uv - \int v\,du$$

$$= x\sin x - \int \sin x\,dx$$

$$= x\sin x - (-\cos x) + C$$

$$= x\sin x + \cos x + C$$

What if we had picked differently

$u = \cos x$, $dv = x\,dx$

$\frac{du}{dx} = -\sin x$, $v = \int x\,dx = \frac{x^2}{2}$

$\frac{du}{dx} = -\sin x \quad , v = \int x \, dx = \frac{x^2}{2}$

$du = -\sin x \, dx$

$\int u \, dv = uv - \int v \, du$

$= \cos x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} (-\sin x) dx$

$= \frac{x^2}{2} \cos x + \int \frac{x^2}{2} \sin x \, dx$